Methods for Solving a Quadratic Equation

Example: $x^2 - 18x - 40 = 0$

You have already learned **two** methods to solve this equation:

Method I: Factorising

$$x^2 - 18x - 40 = 0$$
$$(x - 20)(x + 2) = 0$$
$$x = 20 \vee x = -2$$

Method II: Completing the Square

$$x^2 - 18x - 40 = 0$$
$$(x - 9)^2 - 81 - 40 = 0$$
$$(x - 9)^2 - 121 = 0$$
$$(x - 9)^2 = 121$$
$$x - 9 = 11 \vee x - 9 = -11$$
$$x = 20 \vee x = -2$$

BUT now there is also a THIRD method, called:

THE Quadratic Formula

NEW

How does it work? **Simple!** It's just following this **recipe**

Roadmap ABC formula **Solving a quadratic equation**
1. Make the right hand side zero
2. Put the terms in order
3. Write down the a, the b and the c
4. Calculate the Discriminant $D = b^2 - 4ac$
5. Fill the ABC formula: $x = \frac{-b \pm \sqrt{D}}{2a}$
6. Calculate
7. Answer the question (rounded or exact?)

Method III: The Quadratic Formula

$$x^2 - 18x - 40 = 0$$
$$a = 1 \quad b = -18 \quad c = -40$$

$$D = (-18)^2 - 4 \cdot 1 \cdot -40 = 484$$

$$x = \frac{--18 + \sqrt{484}}{2 \cdot 1} \vee x = \frac{--18 - \sqrt{484}}{2 \cdot 1}$$
$$x = \frac{18 + 22}{2} \vee x = \frac{18 - 22}{2}$$
$$x = \frac{40}{2} \vee x = \frac{-4}{2}$$

$$x = 20 \vee x = -2$$

Proof the Quadratic Formula

Lesson goal

The Quadratic Formula provides us the solutions to an equation of the form $ax^2 + bx + c = 0$ by using $x = \frac{-b+\sqrt{D}}{2a}$ v $x = \frac{-b-\sqrt{D}}{2a}$.

Students are expected to memorize the Quadratic Formula. However, the reasoning behind its validity is not always clear to them. This assignment allows students to derive the Quadratic Formula themselves, providing proof of its validity

Teacher Instructions

- Distribute the worksheet to each student, but keep the instruction steps sheet separate.
- Let students cut out the boxes.
- Encourage students to arrange the steps in the correct order.
    - If students are struggling, offer a small hint.
    - Alternatively, provide the instruction steps sheet to help them figure out the correct order.
- If students finish early, give them the instruction steps sheet to verify their work.
- Discuss the correct order with your students

Also available online:

[Proving the quadratic formula (fisscher.net)]

Proving the Quadratic Formula

Each box shows a step in the proof of the quadratic formula.

- Cut out the boxes
- Arrange the steps in the correct order.
- *Check if the steps are in the correct order*.
- Correct? Glue them into your notebook

$\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$
$x+\frac{b}{2a}=\sqrt{\frac{b^2-4ac}{4a^2}}$ v $x+\frac{b}{2a}=-\sqrt{\frac{b^2-4ac}{4a^2}}$
$\left(x+\frac{b}{2a}\right)^2-\left(\frac{b}{2a}\right)^2+\frac{c}{a}=0$
$\left(x+\frac{b}{2a}\right)^2=\left(\frac{b}{2a}\right)^2-\frac{c}{a}$
$x+\frac{b}{2a}=\frac{\sqrt{b^2-4ac}}{2a}$ v $x+\frac{b}{2a}=-\frac{\sqrt{b^2-4ac}}{2a}$
START $ax^2+bx+c=0$
$\left(x+\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}-\frac{4ac}{4a^2}$
$x=-\frac{b}{2a}+\frac{\sqrt{b^2-4ac}}{2a}$ v $x=-\frac{b}{2a}-\frac{\sqrt{b^2-4ac}}{2a}$
$x^2+\frac{b}{a}x+\frac{c}{a}=0$
FINISH $x=\frac{-b+\sqrt{b^2-4ac}}{2a}$ v $x=\frac{-b-\sqrt{b^2-4ac}}{2a}$

INSTRUCTION STEPS to prove the quadratic formula
1.Divide by a
2.Completing the square
3. Isolate x (part 1)
4. Calculate the square at the right and make the denominators equal
5. Add the fractions
6. Calculate the square root
7. Simplify the square root
8. Isolate x (part 2)
9. Add the fractions

Answer Key

Prove the Quadratic formula

$ax^2 + bx + c = 0$	**STEPS**
$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$	1.Divide by a
$\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a} = 0$	2.Completing the square
$\left(x + \frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}$	3. Isolate x (part 1)
$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{4ac}{4a^2}$	4. Calculate the square at the right and make the denominators equal
$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$	5. Add the fractions
$x + \frac{b}{2a} = \sqrt{\frac{b^2-4ac}{4a^2}}$ v $x + \frac{b}{2a} = -\sqrt{\frac{b^2-4ac}{4a^2}}$	6. Calculate the square root
$x + \frac{b}{2a} = \frac{\sqrt{b^2-4ac}}{2a}$ v $x + \frac{b}{2a} = -\frac{\sqrt{b^2-4ac}}{2a}$	7. Simplify the square root
$x = -\frac{b}{2a} + \frac{\sqrt{b^2-4ac}}{2a}$ v $x = -\frac{b}{2a} - \frac{\sqrt{b^2-4ac}}{2a}$	8. Isolate x (part 2)
$x = \frac{-b+\sqrt{b^2-4ac}}{2a}$ v $x = \frac{-b-\sqrt{b^2-4ac}}{2a}$	9. Add the fractions

Roadmap ABC formula **Solving a quadratic equation**	**Example:** **Solve, by using the ABC formula $15 + 5x = x^2$** **Round to two decimal places.**
1. Make the right hand side zero	1. $-x^2 + 15 + 5x = 0$
2. Put the terms in order	2. $-x^2 + 5x + 15 = 0$
3. Write down the a, the b and the c	3. $a = -1\ ;\ b = 5\ ;\ c = 15$
4. Calculate the Discriminant $D = b^2 - 4ac$	4. $D = 5^2 - 4 \cdot -1 \cdot 15 = 25 + 60 = 85$
5. Fill the ABC formula: $x = \frac{-b \pm \sqrt{D}}{2a}$	5. $x = \frac{-5 \pm \sqrt{85}}{2 \cdot -1}$
6. Calculate	6. $x = \frac{-5 + \sqrt{85}}{2 \cdot -1} \approx -2{,}11$ ∨ $x = \frac{-5 - \sqrt{85}}{2 \cdot -1} \approx 7{,}11$
7. Answer the question (**rounded** or **exact**?)	7. Thus $x \approx -2{,}11$ ∨ $x \approx 7{,}11$
Exercise 1: **Solve, by using the ABC formula $\frac{1}{2}x^2 + 4x = 1$:** **Round to two decimal places.**	**Exercise 2:** **Solve, by using the ABC formula $x + 2x^2 - 5 = 0$** **Round to two decimal places.**
1. Make the right hand side zero: $\frac{1}{2}x^2 + 4x - \_\_\_\_ = 0$	Making the right hand side zero is not necessary!
2. Put the terms in order : $\frac{1}{2}x^2 + 4x - \_\_\_\_ = 0$	Put the terms in order : $2x^2 \_\_\_\_\_ = 0$
3. Write down the a, the b and the c a= $\frac{1}{2}$ b=_____ c=_____	Write down the a, the b and the c : a= _____ b=_____ c=_____
4. Calculate the Discriminant : $D = b^2 - 4ac$ $D = \ldots\ldots^2 - 4 \cdot \frac{1}{2} \cdot \ldots\ldots = \ldots\ldots$	Calculate the Discriminant : $D = b^2 - 4ac$
5. Fill the ABC formula: $x = \frac{-\ldots \pm \sqrt{\ldots\ldots}}{2\ldots\ldots}$	Fill the ABC formula: $x = \frac{-\ldots \pm \sqrt{\ldots\ldots}}{2\ldots\ldots}$
6. Calculate	Calculate
7. Answer the question (rounded or exact?)	Answer the question (rounded or exact?)

Roadmap ABC formula **Solving a quadratic equation**	**Example:** **Solve, by using the ABC formula $15 + 5x = x^2$** **Round to two decimal places.**
8. Make the right hand side zero	1. $-x^2 + 15 + 5x = 0$
9. Put the terms in order	2. $-x^2 + 5x + 15 = 0$
10. Write down the a, the b and the c	3. $a = -1\ ;\ b = 5\ ;\ c = 15$
11. Calculate the Discriminant $D = b^2 - 4ac$	4. $D = 5^2 - 4 \cdot -1 \cdot 15 = 25 + 60 = 85$
12. Fill the ABC formula: $x = \frac{-b \pm \sqrt{D}}{2a}$	5. $x = \frac{-5 \pm \sqrt{85}}{2 \cdot -1}$
13. Calculate	6. $x = \frac{-5 + \sqrt{85}}{2 \cdot -1} \approx -2{,}11 \vee x = \frac{-5 - \sqrt{85}}{2 \cdot -1} \approx 7{,}11$
14. Answer the question (**rounded** or **exact**?)	7. Thus $x \approx -2{,}11 \vee x \approx 7{,}11$

Exercise 1: **Solve, by using the ABC formula $\frac{1}{2}x^2 + 4x = 1$:** **Round to two decimal places.**	**Exercise 2:** **Solve, by using the ABC formula $x + 2x^2 - 5 = 0$** **Round to two decimal places.**
8. Make the right hand side zero: $\frac{1}{2}x^2 + 4x - \underline{\ 1\ } = 0$	Making the right hand side zero is not necessary! ok
9. Put the terms in order : $\frac{1}{2}x^2 + 4x - \underline{\ 1\ } = 0$	Put the terms in order : $2x^2 \underline{+x - 5} = 0$
10. Write down the a, the b and the c a= $\frac{1}{2}$ b= 4 c= −1	Write down the a, the b and the c : a= 2 b= 1 c= −5
11. Calculate the Discriminant : $D = b^2 - 4ac$ $D = 4^2 - 4 \cdot \frac{1}{2} \cdot -1 = 18$	Calculate the Discriminant : $D = b^2 - 4ac$ $D = 1^2 - 4 \cdot 2 \cdot -5 = 41$
12. Fill the ABC formula: $x = \frac{-4 \pm \sqrt{18}}{2 \cdot \frac{1}{2}}$	Fill the ABC formula: $x = \frac{-1 \pm \sqrt{41}}{2 \cdot 2}$
13. Calculate $x = \frac{-4 + \sqrt{18}}{1} \vee x = \frac{-4 - \sqrt{18}}{1}$ $x = -4 + 3\sqrt{2} \vee x = -4 - 3\sqrt{2}$ exact	Calculate $x = \frac{-1 + \sqrt{41}}{4} \vee x = \frac{-1 - \sqrt{41}}{4}$ $x = -\frac{1}{4} + \frac{1}{4}\sqrt{41} \vee x = -\frac{1}{4} - \frac{1}{4}\sqrt{41}$
14. Answer the question (rounded or exact?) $x \approx 0.24 \vee x \approx 8.24$	Answer the question (rounded or exact?) $x \approx 1.35 \vee x \approx -1.85$

$\approx$ approximately equal to

Solving Quadratic Equations: how to choose?

Step 1: Which *mode* does your equation have?

$ax^2 + c = 0$	$ax^2 + bx = 0$	$ax^2 + bx + c = 0$
In this mode $b = 0$	In this mode $c = 0$	In this mode a, b and c all have a value $\neq 0$

Step 2:

$ax^2 = -c$	$ax^2 + bx = 0$	$ax^2 + bx + c = 0$
Try to **simplify** to : $x^2 = -\frac{c}{a}$ **Solve** by taking the **square root**: $x = \sqrt{-\frac{c}{a}} \vee x = -\sqrt{-\frac{c}{a}}$	**Solve** by **factorising the binomial**: $ax(x + \frac{b}{a}) = 0$ $ax = 0 \vee x + \frac{b}{a} = 0$ $x = 0 \vee x = -\frac{b}{a}$	Try to **simplify** to : $x^2 + bx + c = 0$ Go to step 3…

Step 3 (only for $ax^2 + bx + c = 0$)

$x^2 + bx + c = 0$	$x^2 + bx + c = 0$	$ax^2 + bx + c = 0$
Try to solve by **factorising the trinomial**: $(x + \cdots)(x + \cdots) = 0$	Try to solve by **completing the square**	Use the **Quadratic formula**

Step 4

Reread the exercise and answer the question asked.
Points of attention:

- Rounding
- In the case of a context exercise:
  - Full English sentence?
  - Unit